Enrollment No: ______________________ Exam Seat No:______________________

C.U.SHAH UNIVERSITY

Summer Examination-2017

Subject Name: Number Theory

Subject Code:5SC03NTE1 **Branch: M.Sc.(Mathematics)**

Semester: 3 **Date:29/03/2017** **Time:10:30 To 01:30** **Marks:70**

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the Following questions** **(07)**

a. Prove that if p is prime and p/ab, then p/a or p/b. **(02)**

b. Define: Euler function. Find $\phi(360)$. **(02)**

c. Find highest power of 2 that divides 50!. **(02)**

d. Find $\gcd(525, 231)$. **(01)**

Q-2 **Attempt all questions** **(14)**

a. State Chinese remainder theorem. Solve the system of three congruences **(05)**

$$x \equiv 1(mod\ 3), x \equiv 2(mod\ 5), x \equiv 3(mod\ 7)$$

b. In usual notations prove that, $[a, b](a, b) = ab$. **(05)**

c. If p_n is the n^{th} prime numbers, then prove that $p_n \leq 2^{2^{n-1}}, \forall\, n$. **(04)**

OR

Q-2 **Attempt all questions** **(14)**

a. Prove that if $2^k - 1$ is prime $(k > 1)$, then $n = 2^{k-1}(2^k - 1)$ is perfect and every even perfect number is of this form. **(05)**

b. Let $N = a_0 + a_1 10 + a_2 10^2 + \cdots + a_m 10^m$ be the decimal expansion of the positive integer $N, 0 \leq a_k < 10$, and let $S = a_0 + a_1 + \cdots + a_m$. Then prove that $9|N$ if and only if $9|S$. Is 1571724 divisible by 9? Justify. **(05)**

c. Define: Mobious function. Show that Mobious function is multiplicative. **(04)**

Q-3 **Attempt all questions** **(14)**

a. Prove that given integers a and b, with $b > 0$, there exist unique integers q and r satisfying $a = qb + r,\ \ 0 \leq r < b$. **(05)**

b. Prove that if $k > 0$, then $\gcd(ka, kb) = k \gcd(a, b)$. **(03)**

c. Prove that if $a \equiv b \ (mod\ n)$ and $c \equiv d \ (mod\ n)$, then $a + c \ \equiv b + c \ (mod\ n)$ and $ac \equiv bc \ (mod\ n)$. **(03)**

d. Prove that if $n \geq 1$ and $\gcd(a, n) = 1$, then $a^{\phi(n)} \equiv 1 \ (mod\ n)$. **(03)**

OR

Q-3 **Attempt all questions** **(14)**

a. Prove that every positive integer greater than one can be express uniquely as a product of prime, up to the order of the factor. **(05)**

b. Prove that there are infinitely many prime number. **(03)**

c. Show that $\left[\frac{[x]}{n}\right] = \left[\frac{x}{n}\right]$ if n is a positive integer. **(03)**

d. Prove that if p is prime, then$(p-1)! \equiv -1 \ (mod\ p)$. **(03)**

SECTION – II

Q-4 **Attempt the Following questions** **(07)**

a. Express the rational number $\frac{19}{51}$ in finite simple continue fraction. **(02)**

b. Define: Primitive root. Find two primitive roots of 10. **(02)**

c. Determine the infinite continued fraction representation of$\sqrt{7}$. **(02)**

d. Define: Algebraic number. **(01)**

Q-5 **Attempt all questions** **(14)**

a. Solve the linear Diophantine equation $172\ x + 20y = 1000$. **(05)**

b. Prove that the value of any infinite continued fraction is an irrational number. **(05)**

c. Find first three positive solution of the equation $x^2 - 7y^2 = 1$. **(04)**

OR

Q-5 **Attempt all questions** **(14)**

a. If $\frac{p_k}{q_k}$ arethe convergents of the continued fraction expansion of $\sqrt{d}$then, prove that $p_k^2 - dq_k^2 = (-1)^{k+1} t_{k+1}$, where $t_{k+1} > 0, k = 0,1,2,3, \ldots$ **(05)**

b. Prove that the product of two primitive polynomial is primitive. **(05)**

c. Determine the unique irrational number represented by the infinite continued fraction $x = [3; 6, \overline{1,4}]$. **(04)**

Q-6 **Attempt all questions** **(14)**

a. State and prove the Fermat's Last theorem. **(06)**

b. Prove that the k^{th} convergent of the simple continued fraction $[a_0; a_1, a_2, \ldots, a_n]$ has the value $c_k = \frac{p_k}{q_k}, 0 \leq k \leq n$. **(05)**

c. Let θ denote any irrational number. If there is a rational number a/b with $b \geq 1$ such that $\left|\theta - \frac{a}{b}\right| < \frac{1}{2b^2}$, then prove that$a/b$ equals one of the convergents of the simple continued fraction expansion of θ. **(03)**

OR

Q-6 **Attempt all Questions** **(14)**

a. If p is prime and $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,\ a_n$ is incongruent to 0 modulo p, is a polynomial of degree $n \geq 1$ with integral coefficients, then prove that $f(x) \equiv 0 (mod\ p)$ has at most n incongruent solutions modulo p. **(06)**

b. Prove that all the solutions of $x^2 + y^2 = z^2$ with $x, y, z > 0$; satisfying the conditions $(x, y, z) = 1,\ \ 2|x$ are given by the formula $x = 2st, y = s^2 - t^2,\ z = s^2 + t^2$, where $s > t > 0, (s, t) = 1$ and one of s, t is even and the other is odd. **(05)**

c. Compute the convergents of the simple continued fraction $[1; 2,3,3,2,1]$. **(03)**